Inferential Statistics and Probability
a Holistic Approach

Chapter 10

Two Population Inference

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Comparing two population means

- Four models
  - Independent Sampling
    - Known population variances
      - **Two sample Z - test**
    - The 2 population variances are equal
      - **Pooled variance t-test**
    - The 2 population variances are unequal
      - **t-test for unequal variances**
  - Dependent Sampling
      - **Matched Pairs t-test**

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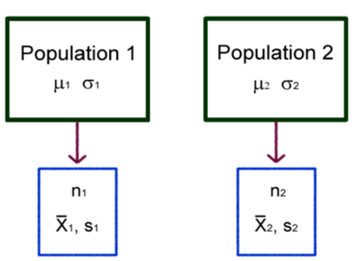
Independent Sampling



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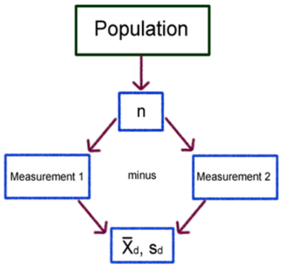
Dependent sampling



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Difference of Two Population means

- $\bar{X}_1 - \bar{X}_2$ is Random Variable
- $\bar{X}_1 - \bar{X}_2$ is a point estimator for $\mu 1 - \mu 2$
- The standard deviation is

  given by the formula $\sqrt{\frac{\sigma_1^{\ 2}}{n_1} + \frac{\sigma_2^{\ 2}}{n_2}}$
- If n1 and n2 are sufficiently large, $\bar{X}_1 - \bar{X}_2$ follows a normal distribution.

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Difference between two means – known population variances

- If both σ_1 and σ_2 are known and the two populations are independently selected, this test can be run.
- Test Statistic:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

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Example 1

- Are larger houses more likely to have pools?
- The housing data square footage (size) was split into two groups by pool (Y/N).
- Can you support the claim that homes with pools have more square feet than the homes without pools. Let $\alpha = .01$

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EXAMPLE 1 – 2 sample Z-test of mean

$$H_o: \mu_1 \le \mu_2 \quad H_a: \mu_1 > \mu_2$$
$$H_o: \mu_1 - \mu_2 \le 0 \quad H_a: \mu_1 - \mu_2 > 0$$

$$\alpha = .01 \qquad Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

H_0 is rejected if $Z > 2.326$ *(or pvalue* $< \alpha$)

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EXAMPLE 1 *Data*

- Population 1
  Size with pool
- Sample size = 130
- Sample mean = 26.25
- Pop Std Dev = 6.93

- Population 2
  Size without pool
- Sample size = 95
- Sample mean = 23.04
- Pop Std Dev = 4.55

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EXAMPLE 1 *DATA*

$$Z = \frac{(26.25 - 23.04) - 0}{\sqrt{\frac{6.93^2}{130} + \frac{4.55^2}{95}}} = 4.19$$

- p-value = P(Z>4.19) = 0.0000137 < 0.01
- Decision: Reject Ho
- Conclusion: Homes with pools have more mean square footage.

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EXAMPLE 1 *p-value method*

- Using Technology Reject Ho if the p-value < α

	Sq ft with pool	*Sq ft no pool*
Mean	26.25	23.04
Std Dev	6.93	4.55
Observations	130	95
Hypothesized Mean Difference	0	
Z	4.19	
p-value	**0.0000137**	

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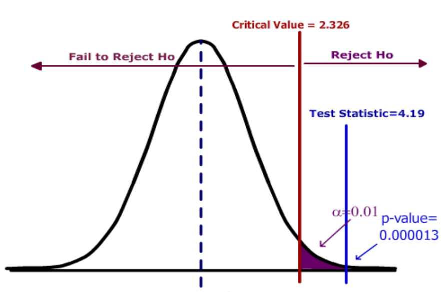
EXAMPLE 1 – Results/Decision



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Pooled variance t-test

- To conduct this test, three assumptions are required:
  - The populations must be normally or approximately normally distributed (or central limit theorem must apply).
  - The sampling of populations must be **independent.**
  - The **population variances** must be **equal**.

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Pooled Sample Variance and Test Statistic

- Pooled Sample Variance:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- Test Statistic:

$$t = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$df = n_1 + n_2 - 2$$

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EXAMPLE 2

- A recent EPA study compared the highway fuel economy of domestic and imported passenger cars.
- A sample of 12 imported cars revealed a mean of 35.76 mpg with a standard deviation of 3.86.
- A sample of 15 domestic cars revealed a mean of 33.59 mpg with a standard deviation of 2.16 mpg.
- At the .05 significance level can the EPA conclude that the mpg is higher on the imported cars? *(Let subscript 2 be associated with domestic cars.)*

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EXAMPLE 2 – critical value method

- : $H_o: \mu_1 \le \mu_2 \quad H_a: \mu_1 > \mu_2$
- : α=.05
- : $t = (\overline{X}_1 - \overline{X}_2)/(s_p\sqrt{1/n_1 + 1/n_2})$
- : H_0 is rejected if t>1.708, df=25
- : t=1.85 H_0 is rejected. Imports have a higher mean mpg than domestic cars.

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t-test when variances are not equal.

- Test statistic: $$t' = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$
- Degrees of freedom: $$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left[\frac{(s_1^2/n_1)^2}{(n_1 - 1)} + \frac{(s_2^2/n_2)^2}{(n_2 - 1)}\right]}$$
- This test (also known as the Welch-Aspin Test) has **less power** then the prior test and should only be used when it is clear the population variances are different.

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EXAMPLE 2

- : $H_o: \mu_1 \le \mu_2 \quad H_a: \mu_1 > \mu_2$
- : α=.05
- : t′ test
- : H_0 is rejected if t>1.746, df=16
- : t'=1.74 H_0 is not rejected. There is insufficient sample evidence to claim a higher mpg on the imported cars.

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Using Technology

- Decision Rule: Reject H_o if pvalue<α
- Megastat: Compare Two Independent Groups
- Use Equal Variance or Unequal Variance Test
- Use Original Data or Summarized Data

domestic 29.8 33.3 34.7 37.4 34.4 32.7 30.2 36.2 35.5 34.6 33.2 35.1 33.6 31.3 31.9

import 39.0 35.1 39.1 32.2 35.6 35.5 40.8 34.7 33.2 29.4 42.3 32.2

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Pooled Variance t-test

- Minitab output
- p-value = 0.038
- p-value < α = .05
- Reject Ho

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Two-sample T for MPG

TYPE     N   Mean  StDev  SE Mean
import  12  35.76   3.86      1.1
US car  15  33.59   2.16     0.56

Difference = μ (import) - μ (US car)
Estimate for difference:  2.16
95% lower bound for difference:  0.16
T-Test of difference = 0 (vs >): T-Value = 1.85
P-Value = 0.038  DF = 25
Both use Pooled StDev = 3.0264
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Unequal Variances t-test

- Minitab output
- p-value = 0.051
- p-value < α = .05
- Fail to Reject Ho

```
Two-sample T for MPG

TYPE     N   Mean  StDev  SE Mean
import  12  35.76   3.86      1.1
US car  15  33.59   2.16     0.56

Difference = μ (import) - μ (US car)
Estimate for difference:  2.16
95% lower bound for difference:  -0.01
T-Test of difference = 0 (vs >): T-Value = 1.74
P-Value = 0.051  DF = 16
```

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Hypothesis Testing - Paired Observations

- Independent samples are samples that are not related in any way.
- Dependent samples are samples that are paired or related in some fashion.
  - For example, if you wished to buy a car you would look at the *same* car at two (or more) *different* dealerships and compare the prices.
- Use the following test when the samples are dependent:

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Hypothesis Testing Involving Paired Observations

$$t = \frac{\bar{X}_d - \mu_d}{s_d / \sqrt{n}}$$

- where $\bar{X}_d$ is the average of the differences
- s_d is the standard deviation of the differences
- n is the number of pairs (differences)

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EXAMPLE 3

- An independent testing agency is comparing the daily rental cost for renting a compact car from Hertz and Avis.
- A random sample of 15 cities is obtained and the following rental information obtained.
- At the .05 significance level can the testing agency conclude that there is a difference in the rental charged?

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Example 3 – continued

- Data for Hertz

$\overline{X}_1 = 46.67$

$s_1 = 5.23$

- Data for Avis

$\overline{X}_2 = 44.87$

$s_2 = 5.62$

City	Hertz	Avis
Atlanta	42	40
Baltimore	51	47
Boston	46	42
Chicago	56	52
Cleveland	45	43
Denver	48	48
Dallas	56	54
Honolulu	37	32
Los Angeles	51	48
Kansas City	45	48
Miami	41	39
New York	44	42
San Francisco	48	45
Seattle	46	50
Washington DC	44	43

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Example 3 - continued

By taking the difference of each pair, variability (measured by standard deviation) is reduced.

$\overline{X}_d = 1.80$

$s_d = 2.513$

$n = 15$

City	Hertz	Avis	Difference
Atlanta	42	40	**2**
Baltimore	51	47	**4**
Boston	46	42	**4**
Chicago	56	52	**4**
Cleveland	45	43	**2**
Denver	48	48	**0**
Dallas	56	54	**2**
Honolulu	37	32	**5**
Los Angeles	51	48	**3**
Kansas City	45	48	**-3**
Miami	41	39	**2**
New York	44	42	**2**
San Francisco	48	45	**3**
Seattle	46	50	**-4**
Washington DC	44	43	**1**

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EXAMPLE 3 *continued*

- $H_0: \mu_d = 0 \qquad H_1: \mu_d \neq 0$
- α=.05
- Matched pairs t test, df=14
- H_0 is rejected if t<-2.145 or t>2.145
- $t = (1.80)/[2.513/\sqrt{15}] = 2.77$
- Reject H_0.
- There is a difference in mean price for compact cars between Hertz and Avis. Avis has lower mean prices.

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Megastat Output – Example 3

Hypothesis Test: Paired Observations

0.000	hypothesized value
46.667	mean Hertz
44.867	mean Avis
1.800	mean difference (Hertz - Avis)
2.513	std. dev.
0.649	std. error
15	n
14	df
2.77	t
.0149	p-value (two-tailed)

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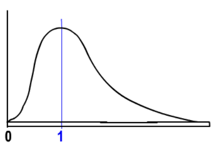
Characteristics of F-Distribution

- There is a "family" of *F* Distributions.
- Each member of the family is determined by two parameters: the numerator degrees of freedom and the denominator degrees of freedom.
- *F* cannot be negative, and it is a continuous distribution.
- The *F* distribution is positively skewed.
- Its values range from 0 *to* ∞. As $F \to \infty$ the curve approaches the X-axis.



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Test for Equal Variances

- For the two tail test, the test statistic is given by:

$$F = \frac{S_i^2}{S_j^2}$$

- s_i^2 *and* s_j^2 are the sample variances for the two populations.
- There are 2 sets of degrees of freedom: n_i-1 for the numerator, n_j-1 for the denominator

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EXAMPLE 4

- A stockbroker at brokerage firm, reported that the mean rate of return on a sample of 10 software stocks was 12.6 percent with a standard deviation of 4.9 percent.
- The mean rate of return on a sample of 8 utility stocks was 10.9 percent with a standard deviation of 3.5 percent.
- At the .05 significance level, can the broker conclude that there is more variation in the software stocks?

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Test Statistic depends on Hypotheses

Hypotheses	Test Statistic
$H_o: \sigma_1 \geq \sigma_2$ $H_a: \sigma_1 < \sigma_2$	$F = \frac{s_2^2}{s_1^2}$ *use* α *table*
$H_o: \sigma_1 \leq \sigma_2$ $H_a: \sigma_1 > \sigma_2$	$F = \frac{s_1^2}{s_2^2}$ *use* α *table*
$H_o: \sigma_1 = \sigma_2$ $H_a: \sigma_1 \neq \sigma_2$	$F = \frac{\max(s_1^2, s_2^2)}{\min(s_1^2, s_2^2)}$ *use* $\alpha/2$ *table*

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EXAMPLE 4 *continued*

- : $H_o: \sigma_1 \leq \sigma_2 \quad H_a: \sigma_1 > \sigma_2$
- : α =.05
- : F-test
- : H_0 is rejected if F>3.68, df=(9,7)
- : F=4.9²/3.5² =1.96 → Fail to Reject H_0.
- There is insufficient evidence to claim more variation in the software stock.

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Minitab Example

- Using Minitab– Test for equal variances under two population independent samples test and click the box to test for equality of variances
- The default p-value is a two-tailed test, so take one-half reported p-value for one-tailed tests
- Example – Domestic vs Import Data
- $H_o: \sigma_1 = \sigma_2 \quad H_a: \sigma_1 \neq \sigma_2$
- α =0.05
- Reject Ho means use unequal variance t-test
- FTR Ho means use pooled variance t-test

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Minitab Output

Test

Null hypothesis $H_0: \sigma_1 / \sigma_2 = 1$
Alternative hypothesis $H_1: \sigma_1 / \sigma_2 \neq 1$
Significance level $\alpha = 0.05$

Method Test

	Statistic	DF1	DF2	P-Value
F	3.20	11	14	0.044

pvalue <0.05
Reject Ho

Use unequal variance t-test to compare means.

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Comparing two proportions

- Suppose we take a sample of n_1 from population 1 and n_2 from population 2.
- Let X_1 be the number of success in sample 1 and X_2 be the number of success in sample 2.
- The sample proportions are then calculated for each group.

$$\hat{p}_1 = \frac{X_1}{n_1} \qquad \hat{p}_2 = \frac{X_2}{n_2}$$

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Hypothesis testing for 2 Proportions

- In conducting a Hypothesis test where the Null hypothesis assumes equal proportions, it is best practice to pool or combine the sample proportions into a single estimated proportion, and use an estimated standard error.

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} \qquad s_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\bar{p}(1-\bar{p})}{n_1} + \frac{\bar{p}(1-\bar{p})}{n_2}}$$

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Hypothesis testing for 2 Proportions

- The test statistic will have a Normal Distribution as long as there are at least 10 successes and 10 failures in both samples.

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\bar{p}(1-\bar{p})}{n_1} + \frac{\bar{p}(1-\bar{p})}{n_2}}}$$

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Example

- In an August 2016 Study, Pew Research asked the sampled Americans if background checks required at gun stores should be made universal extended to all sales of guns between private owners or at gun shows.
- 772 out 990 men said yes, while 857 out of 1020 women said yes.
- Is there a difference in the proportion of men and women who support universal background checks for purchasing guns? Design and conduct the test with a significance level of 1%.

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Example (Design)

- **Ho: $p_m = p_w$** (There is no difference in the proportion of support for background checks by gender)
  Ha: $p_m \neq p_w$ (There is a difference in the proportion of support for background checks by gender)
- **Model:**
  Two proportion Z test. This is a two-tailed test with a = 0.01.
- **Model Assumptions:** for men there are 772 yes and 218 no. For women there are 857 yes and 16 no. Since all these numbers exceed 10, the model is appropriate.
- **Decision Rules:**
  Critical Value Method - Reject Ho if Z > 2.58 or Z < -2.58.
  P-value method - Reject Ho if p-value <0.01

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Example (Results)

$$\hat{p}_m = \frac{772}{990} = 0.780 \qquad \hat{p}_w = \frac{857}{1020} = 0.840$$

$$\bar{p} = \frac{772+857}{990+1020} = 0.810 \qquad Z = \frac{(0.780-0.840)-0}{\sqrt{\frac{0.810(1-0.810)}{990}+\frac{0.810(1-0.810)}{1020}}} = -3.45$$

p-value = 0.0005 < α
Reject Ho Under both methods.

Conclusion: There is a difference in the proportion of support for background checks by gender. Women are more likely to support background checks.

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